Exercise 1.5.5

Assume that the temperature is circularly symmetric: $u = u(r,t)$, where $r^2 = x^2 + y^2$. We will derive the heat equation for this problem. Consider any circular annulus $a \le r \le b$.

(a) Show that the total heat energy is $2\pi \int_a^b c\rho u r\, dr$.

(b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is $-2\pi b K_0 \partial u/\partial r|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Solution

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters a circular annulus at $r = a$, then that same amount must exit at $r = b$ for the temperature to remain the same. If more (less) thermal energy enters at $r = a$ than exits at $r = b$, then the amount of thermal energy in the annulus will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

$$\text{rate of energy in} - \text{rate of energy out} = \text{rate of energy accumulation}$$

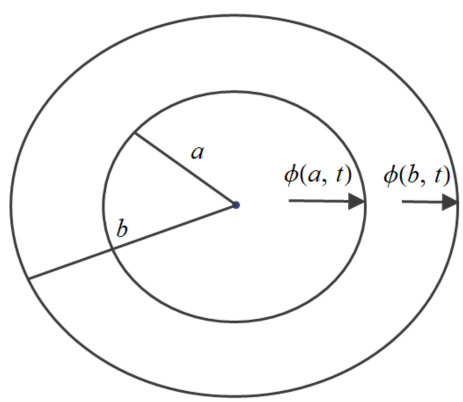

Figure 1: This is a schematic of the circular annulus that the thermal energy flows through. It flows in at $r = a$ and out at $r = b$.

Normally the flux is defined to be the rate that thermal energy flows per unit area, but since we are in two dimensions, it will be the rate that thermal energy flows per unit length. Denote it as $\phi(r,t)$. Multiplying it by the perimeter $P(r)$ that the energy flows through at r gives the rate of energy flow. If we let U represent the amount of energy in the annulus, then the energy balance over it is

$$P(a)\phi(a,t) - P(b)\phi(b,t) = \left.\frac{dU}{dt}\right|_{\text{annulus}}.$$

Factor a minus sign from the left side.

$$-[P(b)\phi(b,t) - P(a)\phi(a,t)] = \left.\frac{dU}{dt}\right|_{\text{annulus}}$$

By the fundamental theorem of calculus, the term in square brackets is an integral.

$$-\int_a^b \frac{\partial}{\partial r}[P(r)\phi(r,t)]\,dr = \left.\frac{dU}{dt}\right|_{\text{annulus}}$$

The thermal energy in the annulus is obtained by integrating the thermal energy density $e(r,t)$ over the annulus's area A.

$$-\int_a^b \frac{\partial}{\partial r}[P(r)\phi(r,t)]\,dr = \frac{d}{dt}\int_A e\,dA$$

For a uniform annulus with mass density ρ, specific heat c, and temperature $u(r,t)$, the thermal energy density is the product $\rho c u$.

$$-\int_a^b \frac{\partial}{\partial r}[P(r)\phi(r,t)]\,dr = \frac{d}{dt}\int_A \rho c u\,dA$$

The area differential for a circle is $dA = P(r)\,dr = 2\pi r\,dr$. The area integral turns into one over the radius.

$$-\int_a^b \frac{\partial}{\partial r}[2\pi r\phi(r,t)]\,dr = \frac{d}{dt}\int_a^b \rho c u(2\pi r\,dr)$$

Therefore, the total thermal energy in the annulus is

$$\boxed{U = 2\pi\int_a^b \rho c u r\,dr.}$$

Divide both sides of the energy balance by 2π and bring the minus sign and time derivative inside the integrals they're in front of.

$$\int_a^b \left\{-\frac{\partial}{\partial r}[r\phi(r,t)]\right\}dr = \int_a^b \rho c\frac{\partial u}{\partial t} r\,dr$$

Since the two integrals are equal over the same interval of integration, the integrands must be equal.

$$-\frac{\partial}{\partial r}[r\phi(r,t)] = \rho c\frac{\partial u}{\partial t} r$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0\frac{\partial u}{\partial r},$$

where K_0 is a proportionality constant known as the thermal conductivity. As a result, the energy balance becomes an equation solely for the temperature.

$$-\frac{\partial}{\partial r}\left(-rK_0\frac{\partial u}{\partial r}\right) = \rho c\frac{\partial u}{\partial t} r$$

Dividing both sides by r, we find that the equation for the temperature is

$$\rho c\frac{\partial u}{\partial t} = \frac{K_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial^2 u}{\partial r^2}\right).$$

Divide both sides by ρc and set $k = K_0/\rho c$ to obtain the circularly symmetric heat equation for a uniform annulus.

$$\boxed{\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)}$$

With Fourier's law in hand, the rate of thermal energy flowing at $r = a$ and $r = b$ can be calculated.

$$\text{Rate of Thermal Energy Flowing at } r = a\text{:} \quad P(a)\phi(a,t) = 2\pi a \left[-K_0 \frac{\partial u}{\partial r}(a,t)\right]$$

$$\text{Rate of Thermal Energy Flowing at } r = b\text{:} \quad P(b)\phi(b,t) = 2\pi b \left[-K_0 \frac{\partial u}{\partial r}(b,t)\right]$$

Therefore,

$$\boxed{\text{Rate of Thermal Energy Flowing at } r = a\text{:} \quad -2\pi a K_0 \frac{\partial u}{\partial r}(a,t)}$$

and

$$\boxed{\text{Rate of Thermal Energy Flowing at } r = b\text{:} \quad -2\pi b K_0 \frac{\partial u}{\partial r}(b,t).}$$